given proceed thus: Take half of the sum of the four given sides, and from it subtract each side in turn. Multiply together the four remainders, and reserve the product. Multiply together the four sides. Take half their product and multiply it by the cosine of the given sum of the angles increased by unity. Regard the sign of the cosine. Subtract this product from the reserved product and take the square root of the remainder. It will be the area of the quadrilateral."]

SOLUTION BY R. J. ADCOCK, MONMOUTH, ILL.

Let a, b, c, d be the four sides of the quadrilateral $ABCD$, $a = AB$, $b = BC$, $c = CD$, $d = AD$, and $\frac{1}{2}(a+b+c+d) = h$, then it is required to show that

$$\text{area } ABCD = \sqrt{[(h-a)(h-b)(h-c)(h-d) - \tfrac{1}{2}abcd(\cos[A+C]+1)]}.$$

Since $BD^2 = a^2 + d^2 + 2ad\cos A = b^2 + c^2 + 2bc\cos C,$ (1)

$$2ad\cos A - 2bc\cos C = b^2 + c^2 - a^2 - d^2,$$

and $a^2d^2\cos^2 A + b^2c^2\cos^2 C - 2abcd\cos A\cos C = \frac{1}{4}(b^2+c^2-a^2-d^2)^2.$ (2)

And since area $= \frac{1}{2}(ad\sin A + bc\sin C)$,

$$a^2d^2\sin^2 A + b^2c^2\sin^2 C + 2abcd\sin A\sin C = 4(\text{area})^2. \quad (3)$$

Adding (2) and (3) and reducing,

$$a^2d^2+b^2c^2 - 2abcd\cos(A+C) = 4(\text{area})^2 + \tfrac{1}{4}(b^2+c^2-a^2-d^2)^2,$$

hence $(\text{area})^2 = \frac{1}{16}[4(a^2d^2+b^2c^2)-(b^2+c^2-a^2-d^2)^2] - \frac{1}{2}abcd\cos(A+C)$

$$= \tfrac{1}{16}[4a^2d^2+4b^2c^2-(b^2+c^2-a^2-d^2)^2+8abcd] - \tfrac{1}{2}abcd[1+\cos(A+C)]$$

$$= (h-a)(h-b)(h-c)(h-d) - \tfrac{1}{2}abcd[1+\cos(A+C)],$$ as required.

SOLUTIONS OF PROBLFMS IN NUMBER FOUR.

SOLUTIONS of problems in number 4 have been received as follows;

From R. J. Adcock, 173 and answer to Mr. Baker's query; Henry Gunder, 171; Henry Heaton, 170, 171 and 173; Prof E. W. Hyde, 171; G. W. Hill, 170; Chas. H. Kummell, 170, 171 and 174; Artemas Martin, 172; E. B. Seitz, 170, 171 and 173.

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170. "Given the lengths of the eight edges of a quadrangular pyramid to find its altitude."

SOLUTION BY G. W. HILL, PH. D., NYACK. TURNPIKE, N. Y.

Denoting the solid angles by the symbols 0, 1, 2, 3, 4, of which 0 belongs to the vertex, and the edges by (01), (02), &c., let us divide the pyramid into two triangular pyramids by a plane passing through 0, 1, 3. Denote six time the volumes of the pyramids 0123 and 0134 severally by Δ and Δ', and twice the areas of their bases by A and A', and by x their common altitude. Then we shall have

$$x^2 = \frac{\Delta^2}{A^2} = \frac{\Delta'^2}{A'^2},$$

whence

$$A'^2 \Delta^2 = A^2 \Delta'^2. \tag{1}$$

If we employ the notation

$$[12] = \frac{(01)+(02)-(12)}{2},\quad [23] = \frac{(02)+(03)-(23)}{2}, \text{ \&c.,}$$

and, in order to distinguish it as an unknown quantity, put y for [13], we have for Δ^2, Δ'^2, A^2 and A'^2 the following expressions

$$\Delta^2 = (01)(02)(03) - (01)[23]^2 - (03)[12]^2 + 2[12][23]y - (02)y^2,$$
$$\Delta'^2 = (01)(03)(04) - (01)[34]^2 - (03)[14]^2 + 2[14][34]y - (04)y^2,$$
$$A^2 = \Big[(01)+(02)-2[12]\Big]\Big[(02)+(03)-2[23]\Big] - \Big[(02)-[12]-[23]+y\Big]^2,$$
$$A'^2 = \Big[(03)+(04)-2[34]\Big]\Big[(01)+(04)-2[14]\Big] - \Big[(04)-[14]-[34]+y\Big]^2.$$

For the proof of these equations see a memoir of Lagrange, *Solutions analytiques de quelques probleˋmes sur les Pyramides Triangulaires.* Tome III, p. 659.

On the substitution of these values in (1), we have an equation of the fourth degree in y, which serves to determine this quantity and thence x.

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171. "A pail in the form of a frustrum of a cone—radius of upper base $2r$, of lower base r, and height h—is inclined so that if water be poured in to it, the water will just reach the lower edge of the upper base, and the upper edge of the lower base. How many gallons of water in the pail?"

SOLUTION BY PROF. E. W. HYDE, UNIV. OF CINCINNATI, CIN., OHIO.

Let $ABCD$ (see fig. on next page) be the pail and AC the surface of the water, then we have to find the volume $ABC = AVC - AVB$. Volume $AVC =$ (area of ellipse AC)$\times\frac{1}{3}p$, in which $p = VE =$ perp. from V on AC

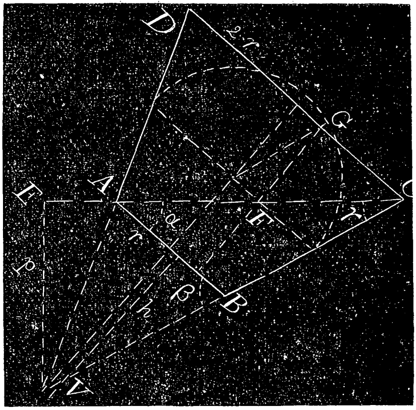
Volume of $AVB = \frac{1}{3}h\times\pi r^2 = \frac{1}{3}\pi r^2 h$.

From the figure $\alpha = \sin^{-1}\dfrac{h}{(h^2+9r^2)^{1/2}}$ $= \cos^{-1}\dfrac{3r}{(h^2+9r^2)^{1/2}}$, $\beta = \sin^{-1}\dfrac{h}{(h^2+r^2)^{1/2}}$ $= \cos^{-1}\dfrac{r}{(h^2+r^2)^{1/2}}$, and $\gamma = \beta - \alpha$

$\therefore p = VC\sin\gamma = 2\sqrt{(h^2+r^2)}.\sin(\beta+\alpha)$

$= 2\sqrt{(h^2+r^2)}\left[\dfrac{3hr-hr}{\sqrt{(h^2+9r^2)}\sqrt{(h^2+r^2)}}\right]$

$= \dfrac{4hr}{\sqrt{(h^2+9r^2)}}$.



Let a and b be the semi axes of the ellipse AC; then its area is πab. $a = \frac{1}{2}AC = \frac{1}{2}\sqrt{(h+9r^2)}$, and $b = FG = \sqrt{[(\frac{3}{2}r)^2-(\frac{1}{2}r)^2]} = r\sqrt{2}$; $\pi ab = \frac{1}{2}\pi r\sqrt{2}\times\sqrt{(h^2+9r^2)}$.

$$\therefore \text{ Vol. } AVC = \frac{4hr}{3\sqrt{(h^2+9r^2)}}\cdot\tfrac{1}{2}\pi r\sqrt{2}\times\sqrt{(h^2+9r^2)} = \tfrac{2}{3}\sqrt{2}\pi hr^2.$$

Hence Vol. $ABC = \frac{2}{3}\sqrt{2}\times\pi hr^2 - \frac{1}{3}\pi hr^2 = \frac{1}{3}(2\sqrt{2}-1).\pi hr^2$.

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172. "Find the least integral values of x and y that will satisfy the equation $x^2 - 9817y^2 = 1$."

SOLUTION BY ARTEMAS MARTIN, M. A., ERIE, PA.

Put $A = 9817$, then $\sqrt{A} = \sqrt{(9817)} = r + \cfrac{1}{u_1 + \cfrac{1}{u_2 + \cfrac{1}{u_3 + \text{ etc.,}}}}$

where r is the integral part of $\sqrt{A}$.

The last quotient of every complete period is $2r$. Let m be the number of quotients in a complete period, and $p_m \div q_m$ the last convergent in the first period; then, when m is even, $x = p_m$, $y = q_m$, and when m is odd, $x = p_{2m}$, $y = q_{2m}$.

Let $(r + a_n) \div b_n = u_n +$ etc. and $(r + a_{n+1}) \div b_{n+1} = u_{n+1} +$ etc. be any two consecutive complete quotients, then $a_0 = 0$, $b_0 = 1$; $a_1 = r$, $b_1 = A - r^2$; $u_0 = r$, $u_1 = 2r \div (A - r^2)$; $a_{n+1} = u_n b_n - a_n$, $b_{n+1} = (A - a^2_{n+1}) \div b_n$.

If $p_n \div q_n$, $p_{n+1} \div q_{n+1}$ be any two consecutive convergents and u_{n+1} the quotient corresponding to $p_{n+1} \div q_{n+1}$, then

$$\frac{p_1}{q_1}=\frac{r}{1},\ \frac{p_2}{q_2}=\frac{ru_1+1}{u_1},\ \ldots\ \frac{p_{n+2}}{q_{n+2}}=\frac{u_{n+1}\,p_{n+1}+p_n}{u_{n+1}\,q_{n+1}+q_n}.$$

The partial quotients are easily found to be

99; 12, 2, 1, 1, 1, 2, 2, 7, 1, 5, 8, 11, 1, 1, 6, 1, 4, 2, 21, 1, 1, 3, 2, 1, 2, 17, 1, 1, 1, 4, 5, 1, 3, 1, 3, 2, 1, 65, 2, 1, 3, 2, 5, 1, 3, 24, 1, 1, 24, 3, 1, 5, 2, 3, 1, 2, 65, 1, 2, 3, 1, 3, 1, 5, 4, 1, 1, 1, 17, 2, 1, 2, 3, 1, 1, 21, 2, 4, 1, 6, 1, 1, 11, 8, 5, 1, 7, 2, 2, 1, 1, 1, 2, 12, 198 $= u_{95} = 2r$.

As 95, the number of quotients in a period, is odd, therefore $x = p_{95}$, $y = q_{95}$ satisfy the equation $x^2 - 9817y^2 = -1$; and $x = p_{190}$, $y = q_{190}$ satisfy the equation $x^2 - 9817y^2 = +1$.

It is not necessary to compute the *numerators* of the convergent fractions as $p_m = rq_m + q_{m-1}$ Computing the values of q_1, q_2, q_3, etc., we find

$$q_{95} = 7441734799204446071122530309232151170560356045,$$
$$p_{95} = 737332852203490945759241163585096785868514690632.$$

$x = p_{190} = 2p_{95}^2+1 = 108731946987707004565417150001997268987807895584585116579452204181943260442884680816719733711 8849,$

$y = q_{190} = 2p_{95}q_{95} = 1097407108967877441016107896323307015642289401035150681407653671863307274550379924301389214088 0.$

These numbers are believed to be the largest of the kind that have yet been found.

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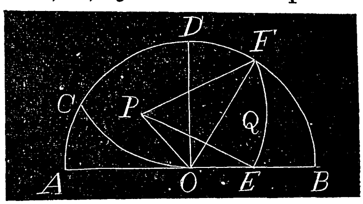
173. "A Given semicircle is divided into two quadrants, and a point is taken at random in each quadrant; find the chance that the distance between them is less than the radius of the semicircle."

SOLUTION BY E. B. SEITZ, GREENVILLE, OHIO.

Let AOD and BOD be the two quadrants, P, Q the random points. With D, P as centers and radii each equal to the radius of the semicircle, describe the arcs OC, EF. Join OP, OF, PF, PE.



If P is in the surface OCD, and Q in the surface $OEFD$, the distance between them is less than the radius of the semicircle.

Put $OA = 1$, $OP = x$, $\angle POD = \theta$, and area $OEFD = u$. Then

$$u = \tfrac{1}{2}\sin^{-1}(x\cos\theta)+\cos^{-1}(\tfrac{1}{2}x)-\tfrac{1}{4}\pi-\tfrac{1}{4}x\sqrt{(4-x^2)}+\tfrac{1}{2}x\cos\theta\sqrt{(1-x^2\cos^2\theta)}$$
$$-\tfrac{1}{2}x^2\sin\theta\cos\theta.$$

But when P is in the surface OAC, the arc described from it does not cut the arc BD, and the area cut off from the quadrant BOD is

$$u_1 = \tfrac{1}{2}\sin^{-1}(x\cos\theta)+\tfrac{1}{2}\cos^{-1}(x\sin\theta)+\tfrac{1}{2}x\cos\theta\sqrt{(1-x^2\cos^2\theta)}$$
$$-\tfrac{1}{2}x\sin\theta\sqrt{(1-x^2\sin^2\theta)}-x^2\sin\theta\cos\theta.$$

Hence, since the whole number of ways the two points can be taken is $\frac{1}{16}\pi^2$, we have for the required chance

$$p=\frac{16}{\pi^2}\int_0^{\frac{1}{3}\pi}\int_0^1 uxdxd\theta+\frac{16}{\pi^2}\int_{\frac{1}{3}\pi}^{\frac{1}{2}\pi}\left[\int_0^{2\cos\theta}uxdx+\int_{2\cos\theta}^1 u_1xdx\right]d\theta$$

$$=\frac{1}{2\pi^2}\int_0^{\frac{1}{3}\pi}[8\pi-8\theta-\pi\sec^2\theta+2\theta\sec^2\theta+2\tan\theta-6\sqrt{3}]d\theta$$

$$+\frac{1}{2\pi^2}\int_{\frac{1}{3}\pi}^{\frac{1}{2}\pi}[16\pi-32\theta-\pi\sec^2\theta-2\pi\operatorname{cosec}^2\theta+2\theta\sec^2\theta+2\tan\theta$$
$$+6\theta\operatorname{cosec}^2\theta-6\cot\theta-80\sin\theta\cos\theta+128\sin^3\theta\cos\theta-64\sin^5\theta\cos\theta$$
$$+64\sin\theta\,\mathrm{sos}^5\theta]d\theta$$

$$=\frac{4}{3}-\frac{\sqrt{3}}{\pi}-\frac{2}{\pi^2}.$$

[Mr. Heaton, by a similar process, gets $p=\frac{4}{3}-\frac{\sqrt{3}}{\pi}-\frac{1}{\pi^2}$; and Mr. Adcock, by restricting the random points to the periphery of the quadrants, gets $p=\frac{4}{9}$. We have not verified the operations in either Mr. Seitz' or Mr. Heaton's solution, and therefore cannot say which is correct.]

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174. "Give the most convenient method to compute $\Gamma(\frac{1}{6}n)$, n being an integer."

SOLUTION BY CHAS. H. KUMMELL, DETROIT, MICH.

Because $\Gamma(\frac{1}{6}n)=(\frac{1}{6}n-1)\Gamma(\frac{1}{6}n-1)=(\frac{1}{6}n-1)(\frac{1}{6}n-2)\ldots(\frac{1}{6}n-r)\Gamma(\frac{1}{6}n-r)$, (1)
where $1>\frac{1}{6}n-r>0$, it is only necessary to determine

$$\Gamma(\tfrac{1}{6}),\ \Gamma(\tfrac{2}{6}),\ \Gamma(\tfrac{3}{6}),\ \Gamma(\tfrac{4}{6}),\ \Gamma(\tfrac{5}{6}).$$

Of these $\Gamma(\frac{3}{6})=\Gamma(\frac{1}{2})=\sqrt{\pi}$ is known.

By the theorem:

$$\Gamma(n)\,\Gamma(1-n)=\pi\operatorname{cosec}n\pi \tag{2}$$

we have

$$\Gamma(\tfrac{1}{6})\,\Gamma(\tfrac{5}{6})=\pi\operatorname{cosec}\tfrac{1}{6}\pi=2\pi,\ \therefore\ \Gamma(\tfrac{5}{6})=2\pi\div\Gamma(\tfrac{1}{6}), \tag{3}$$

$$\Gamma(\tfrac{2}{6})\,\Gamma(\tfrac{4}{6})=\pi\operatorname{cosec}\tfrac{1}{3}\pi=2.3^{-\frac{1}{2}}\pi,\ \therefore\ \Gamma(\tfrac{2}{3})=2.3^{-\frac{1}{2}}.\pi\div\Gamma(\tfrac{1}{3}). \tag{4}$$

It is then only necessary to compute $\Gamma(\frac{1}{6})$ and $\Gamma(\frac{1}{3})$. By Gauss' theorem:

$$\Gamma(n)\,\Gamma\left(n+\frac{1}{r}\right)\Gamma\left(n+\frac{2}{r}\right)\ldots\Gamma\left(n+\frac{r-1}{r}\right)=(2\pi)^{\frac{r-1}{2}}r^{-rn+\frac{1}{2}}\Gamma(nr), \tag{5}$$

we have, placing $n=\frac{1}{6}$ and $r=2$, $\Gamma(\frac{1}{6})\,\Gamma(\frac{4}{6})=2^{\frac{2}{3}}\pi^{\frac{1}{2}}\,\Gamma(\frac{2}{6})$; hence by (4)

$$\Gamma(\tfrac{1}{6})=2^{-\frac{1}{3}}3^{\frac{1}{2}}\pi^{-\frac{1}{2}}\,\Gamma(\tfrac{1}{3})^2. \tag{6}$$

If then $\Gamma(\frac{1}{3})$ is known $\Gamma(\frac{1}{6}n)$ is known.

To compute this we employ the definite integral

$$u = \int_0^{\frac{1}{2}\pi} \frac{d\varphi}{(\sin\varphi)^{\frac{2}{3}}} = \tfrac{1}{2}B(\tfrac{1}{6}, \tfrac{1}{2}) = \frac{\Gamma(\frac{1}{6})\,\Gamma(\frac{1}{2})}{2\Gamma(\frac{2}{3})}$$

$$= 2^{-2}3^{\frac{1}{2}}\pi^{-\frac{1}{2}}\Gamma(\tfrac{1}{6})\,\Gamma(\tfrac{1}{3}) \text{ by (4)}, = 2^{-\frac{7}{3}}\,3\,\pi^{-1}\Gamma(\tfrac{1}{3})^3 \text{ by (6)}.$$

Therefore
$$\Gamma(\tfrac{1}{3}) = 2^{\frac{7}{9}}3^{-\frac{1}{3}}(u\pi)^{\frac{1}{3}}. \tag{7}$$

The integral u may be evaluated as follows:

Place
$$\sin\varphi = (1+\sqrt{3}.\cot^2\tfrac{1}{2}\psi)^{-\frac{3}{2}},$$

$$\therefore \quad \cos\varphi\, d\varphi = \frac{3^{\frac{3}{2}}\cot\frac{1}{2}\psi\, d\psi}{2\sin^2\frac{1}{2}\psi(1+\sqrt{3}.\cot^2\frac{1}{2}\psi)^{\frac{5}{2}}} = \frac{3^{\frac{3}{2}}\cot\frac{1}{2}\psi d\psi}{2\sin^2\frac{1}{2}\psi}(\sin\varphi)^{\frac{5}{3}}$$

$$\cos\varphi = \sin\varphi\,\sqrt{[3\,\sqrt{3}(\cot^2\tfrac{1}{2}\psi + \sqrt{3}.\cot^4\tfrac{1}{2}\psi + \cot^6\tfrac{1}{2}\psi)]}$$

$$= 3^{\frac{3}{4}}\sin\varphi\cot\tfrac{1}{2}\psi\,\frac{\sqrt{[\sin^4\frac{1}{2}\psi+\sqrt{3}.\sin^2\frac{1}{2}\psi\cos^2\frac{1}{2}\psi+\cos^4\frac{1}{2}\psi}}{\sin^2\frac{1}{2}\psi}$$

$$= 2^{-1}3^{\frac{3}{4}}\sin\varphi\,\frac{\cot\frac{1}{2}\psi}{\sin^2\frac{1}{2}\psi}\,\sqrt{[(1-\cos\psi)^2+\sqrt{3}.\sin^2\psi+(1+\cos\psi)^2]}$$

$$= 2^{-1}3^{\frac{3}{4}}\sin\varphi\,\frac{\cot\frac{1}{2}\psi}{\sin^2\frac{1}{2}\psi}\,\sqrt{[2^2\cos^2\psi+(\sqrt{\tfrac{1}{2}}+\sqrt{\tfrac{3}{2}})^2\sin^2\psi]};$$

therefore, since $\psi = 0$ if $\varphi = 0$, and $\psi = \pi$ if $\varphi = \frac{1}{2}\pi$,

$$u = \int_0^{\frac{1}{2}\pi}\frac{d\varphi}{(\sin\varphi)^{\frac{2}{3}}} = \int_0^{\pi}\frac{3^{\frac{3}{4}}d\psi}{\sqrt{[2^2\cos^2\psi+(\sqrt{\frac{1}{2}}+\sqrt{\frac{3}{2}})^2\sin^2\psi]}} \tag{8}$$

Comparing this with form (9) ANALYST, Vol. IV, page 121, we have

$$a_0 = 2;\ b_0 = \sqrt{\tfrac{1}{2}}+\sqrt{\tfrac{3}{2}} = 1.93185165;\ \psi_0 = \pi.$$

The arithmetic-geometric mean denoted there by b_n might be better denoted thus

$$\text{arithmetic-geometric mean of } a_0 \text{ and } b_0 = \left\|\tfrac{1}{2}\left\{a_0 \overset{+}{\underset{\times}{\,}} b_0\right\}^{\frac{1}{2}}\right\|.$$

In this case we have $\left\|\frac{1}{2}\left\{2 \overset{+}{\underset{\times}{\,}} (\sqrt{\frac{1}{2}}+\sqrt{\frac{3}{2}})\right\}^{\frac{1}{2}}\right\| = 1.96577817,$

and
$$\psi_1 = 2\pi,\ \psi_2 = 2^2\pi, \ldots\ \psi_n = 2^n\pi.$$

Applying formula (13), ibid., we have

$$u = \frac{3^{\frac{3}{4}}\psi_n}{2^n\left\|\frac{1}{2}\left\{a_0 \overset{+}{\underset{\times}{\,}} b_0\right\}^{\frac{1}{2}}\right\|} = \frac{3^{\frac{3}{4}}\pi}{\left\|\frac{1}{2}\left\{2 \overset{+}{\underset{\times}{\,}} (\sqrt{\frac{1}{2}}+\sqrt{\frac{3}{2}})\right\}^{\frac{1}{2}}\right\|}. \tag{9}$$

We have then by (7)

$$\Gamma(\tfrac{1}{3}) = 2^{\frac{7}{9}}3^{-\frac{1}{12}}\pi^{\frac{2}{3}}\left\|\tfrac{1}{2}\left\{2 \overset{+}{\underset{\times}{\,}} (\sqrt{\tfrac{1}{2}}+\sqrt{\tfrac{3}{2}})\right\}^{\frac{1}{2}}\right\|^{-\frac{1}{3}}. \tag{10}$$

This together with (1) (3) (4) and (6) gives $\Gamma(\frac{1}{6}n)$.